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Quotient Star Bodies, Intersection Bodies, and Star Duality

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INTRODUCTION

Various topological constructions have their metric counterparts; for example, metric product corresponds to topological product (see [6]) and metric inverse limit corresponds to topological inverse limit (see [9, 10]). The main subject of this paper is a metric counterpart of the topological notion of quotient space (see [2] or [3]). We restrict our consideration to a compact metric space A and a very simple equivalence relation, with only one nontrivial equivalence class $C \subset A$. Then, evidently, the natural quotient map $p\colon A \to A/C$ (defined by $p(x) = [x]$) satisfies the condition

$$p(C) \text{ is a singleton and } p|(A \setminus C) \text{ is a topological embedding.} \tag{0.1}$$

Topologists used to look at quotient space "up to a homeomorphism"; i.e., more precisely, they work with the category Top of topological spaces and continuous maps. For geometers, it is important to choose a "good" representative of the topological type of A/C; this representative will be the image of A under a map p_C satisfying (0.1), with some nice geometric properties.

We shall keep in mind the following physical interpretation. Let A be a soft body with a stone C inside. When the stone is cut out, the hole shrinks to a point and the remaining part $A \setminus C$ changes its geometric shape. The shape of the resulting body A/C depends not only on geoemtric properties of A and C, but also on physical properties of A and forces acting on A.

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This interpretation justifies further restrictions. We assume that A is a subset of R^n, $0 \in C$, and $A \setminus C$ is homeomorphic to $A \setminus \{0\}$. Then A/C is homeomorphic to a subset of R^n and so we can look for a map p_C satisfying (0.1), with values in R^n. Moreover, if $p_C\colon \mathrm{R}^n \to \mathrm{R}^n$ satisfying (0.1) is independent of A, then it is reasonable to consider properties of A invariant under p_C.

Following [3], we shall refer to p_C as a *quotient map* (an *identification* in terminology of [2]).

The following theorem follows directly from [2, Theorem 7.7, p. 17 and Theorem 4.3, p. 126].

THEOREM 0.1. *Let* $\mathscr{C}$ *be a nonempty family of compact subsets of* R^n *with* $0 \in C$ *and* $\mathrm{R}^n \setminus C$ *homeomorphic to* $\mathrm{R}^n \setminus \{0\}$ *for every* $C \in \mathscr{C}$, *and let* p_C *be a quotient map with* $p_C(C) = 0$. *For every map* $f\colon \mathrm{R}^n \to \mathrm{R}^n$ *preserving* $\mathscr{C}$, *with* $f(0) = 0$, *and for every* $C \in \mathscr{C}$, *let*

$$\hat{f}(x) := p_{f(C)} f p_C^{-1}(x) \quad \text{for every } x \in \mathrm{R}^n. \tag{0.2}$$

Then

(i) *the formula* (0.2) *defines a map* $\hat{f}\colon \mathrm{R}^n \to \mathrm{R}^n$ *such that* $\hat{f}(0) = 0$ *and the diagram*

$$\begin{array}{ccc} \mathrm{R}^n & \xrightarrow{f} & \mathrm{R}^n \\ {\scriptstyle p_C}\downarrow & & \downarrow{\scriptstyle p_{f(C)}} \\ \mathrm{R}^n & \xrightarrow[\hat{f}]{} & \mathrm{R}^n \end{array}$$

commutes, and

(ii) *if* f *is a homeomorphism, then so is* $\hat{f}$.

We shall work with the class of star bodies in R^n with nonnegative and continuous radial maps. Section 1 concerns star bodies and star maps. In Section 2 we define radial quotient maps and study corresponding quotient bodies. In particular, we prove that the operation $(A, C) \mapsto A/C$ preserves the class of intersection bodies of star bodies (Corollary 2.8).

Section 3 deals with the category $\mathbf{St}^n$ of star bodies and star maps, and its subcategory $\mathbf{St}^n_+$ with star bodies whose radial functions are positive. We define and study a duality on this subcategory: a functor from $\mathbf{St}^n_+$ to itself which is an involution (compare [11]).

We use the following notation: For affine independent points $x_0, \dots, x_k$ in R^n, the simplex with vertices $x_0, \dots, x_k$ is denoted by $\Delta(x_0, \dots, x_k)$; in particular, if a, b are distinct points, then $\Delta(a, b)$ is the segment with endpoints a, b.

As usual, B^n and S^{n-1} are the unit ball and the unit sphere, and κ_n is the volume of B^n. We denote by σ the spherical Lebesgue measure on S^{n-1}. For any $u \in S^{n-1}$, the (linear) hyperplane orthogonal to u is denoted by $u^{\perp}$. For any convex body $A \subset R^n$, its support function is $h(A, \cdot)$, supporting hyperplane is $H(A, \cdot)$, and $b(A)$ is the mean width of A. The class of convex bodies in R^n is $\mathscr{K}_0^n$.

For basic notions of the category theory, see [8].

1. STAR BODIES AND STAR MAPS

Since in the literature there are various notions of a star body (see [4, p. 18; [12], p. 416), we start from definitions: Let A be a nonempty compact subset of R^n and $a \in A$. Then, A is a *body* if and only if $A = \operatorname{cl} \operatorname{int} A$, and A is *star shaped at a* if and only if $\Delta(a, x) \subset A$ for every $x \in A$.

For convenience, we shall assume that $a = 0$. The *radial function* ϱ_A: $\mathrm{R}^n \setminus \{0\} \to \mathrm{R}$ of a set A which is star shaped at 0 is defined by the formula

$$\varrho_A(x) := \sup\{\lambda \geq 0 \mid \lambda x \in A\}.$$

(Sometimes, we write $\varrho(A, x)$ instead of $\varrho_A(x)$.) Obviously, ϱ_A is homogeneous of degree -1.

A set $A \subset \mathrm{R}^n$ will be called a *star body* whenever A is a body which is star shaped at 0 and whose radial function restricted to S^{n-1} is continuous.

Let us notice that this notion of star body is less general than that of Gardner [4], since we assume that $0 \in A$ and thus $\varrho_A \geq 0$. Moreover, Gardner assumes only continuity of the restriction of radial function to its support. On the other hand, our notion is more general than that of Schneider [12], since we do not require $\varrho_A > 0$; that is, we allow 0 to belong to bd A.

Let $\mathscr{S}_0^n$ be the class of all the star bodies in R^n and let

$$\mathscr{S}_1^n = \mathscr{S}_0^n \cup \{\{0\}\}.$$

We are looking for a possibly large group of transformation which preserve $\mathscr{S}_0^n$ and $\mathscr{S}_1^n$.

DEFINITION 1.1. A map f: $\mathrm{R}^n \to \mathrm{R}^n$ is a *star map* if and only if f is a positively homogeneous homeomorphism.

Of course, the set of all the star maps of R^n is a group of transformations. We shall denote it by $\mathrm{GS}(n)$ (general star maps).

The following is evident.

PROPOSITION 1.2. *If $f \in \mathrm{GS}(n)$ and $A \in \mathscr{S}_0^n$, then for every $x \in \mathrm{R}^n$,*

$$\varrho_{f(A)}(f(x)) = \varrho_A(x).$$

The next statement is a direct consequence of Proposition 1.2.

PROPOSITION 1.3. *The classes $\mathscr{S}_0^n$ and $\mathscr{S}_1^n$ are invariant under star maps.*

Let us prove the following.

PROPOSITION 1.4. *Every two star bodies A_1, A_2 with 0 in the interior are star equivalent; i.e., there exists $g \in \mathrm{GS}(n)$ such that $g(A_1) = A_2$.*

Proof. Let $A_1, A_2 \in \mathscr{S}_0^n$. We define $g\colon \mathrm{R}^n \to \mathrm{R}^n$ by the formula

$$g(x) = \begin{cases} \dfrac{\varrho_{A_2}(x)}{\varrho_{A_1}(x)} \cdot x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{1.1}$$

It is clear that $g \in \mathrm{GS}(n)$.

Let us show that $g(A_1) = A_2$. Indeed, g is a homeomorphism and maps $\operatorname{bd} A_1$ onto $\operatorname{bd} A_2$:

$$\begin{aligned} x \in \operatorname{bd} A_1 \quad &\Leftrightarrow \quad \varrho_{A_1}(x) = 1 \quad \Leftrightarrow \quad g(x) = \varrho_{A_2}(x) \cdot x \\ &\Leftrightarrow \quad \varrho_{A_2}(g(x)) = 1 \quad \Leftrightarrow \quad g(x) \in \operatorname{bd} A_2. \end{aligned}$$

∎

Evidently, $\mathrm{GL}(n)$ is a subgroup of $\mathrm{GS}(n)$. As a consequence of Proposition 1.4, we obtain the following corollary:

COROLLARY 1.5. $\mathrm{GL}(n)$ *is a proper subgroup of* $\mathrm{GS}(n)$.

Let us notice that the group $\mathrm{GS}(n)$ is generated by its two subgroups,

$$\mathrm{SS}(n) := \{f \in \mathrm{GS}(n) \mid f(S^{n-1}) = S^{n-1}\} \tag{1.2}$$

and

$$\mathrm{RS}(n) := \{f \in \mathrm{GS}(n) \mid \exists \lambda\colon \mathrm{R}^n \to \mathrm{R}_+,\ f(x) = \lambda(x) \cdot x \text{ for every } x\}. \tag{1.3}$$

The first subgroup consists of *spherical star maps*, the star extensions of homeomorphisms of S^{n-1} onto itself; the second subgroup consists of *radial star maps*, the star extensions of central projections of S^{n-1}.

PROPOSITION 1.6. *For every $f \in \mathrm{GS}(n)$ there exists $g \in \mathrm{RS}(n)$ and $h \in \mathrm{SS}(n)$ such that $f = gh$. This decomposition is unique.*

Proof. Let $f \in \mathrm{GS}(n)$ and let $A = f(B^n)$. Then A is a star body with $0 \in \operatorname{int} A$, and thus, by Proposition 1.4, there exists $g \in \mathrm{GS}(n)$ such that $g(S^{n-1}) = \operatorname{bd} A$. The map g is defined by (1.1) for $A_1 = B^n$ and $A_2 = A$, whence $g \in \mathrm{RS}(n)$.

Let

$$h := g^{-1} f.$$

Then h is a star map preserving S^{n-1}, and $f = gh$.

To prove the uniqueness of the decomposition of f, suppose that

$$f = g_1 h_1 = g_2 h_2$$

for some radial star maps g_1, g_2 and spherical star maps h_1, h_2. Then $g_2^{-1} g_1 = h_2 h_1^{-1}$, where the left side belongs to $\mathrm{RS}(n)$ and the right side belongs to $\mathrm{SS}(n)$. Since these two subgroups have only the identity map in common, it follows that $g_1 = g_2$ and $h_1 = h_2$. ∎

Every map $f\colon \mathrm{R}^n \to \mathrm{R}^n$ preserving $\mathscr{S}_1^n$ induces the map $f_*\colon \mathscr{S}_1^n \to \mathscr{S}_1^n$ defined by

$$f_*\colon A \mapsto f(A).$$

According to [4], the radial metric δ in $\mathscr{S}_1^n$ is defined by the formula

$$\delta(A_1, A_2) = \sup_{u \in S^{n-1}} \left| \varrho_{A_1}(u) - \varrho_{A_2}(u) \right|.$$

THEOREM 1.7. *If $f \in \mathrm{GS}(n)$, then f_* is a Lipschitz map with respect to the radial metric δ. If, moreover, $f \in \mathrm{SS}(n)$, then f_* is an isometry.*

Proof. Let $A_1, A_2 \in \mathscr{S}_1^n$. Then, by Proposition 1.2,

$$\begin{aligned}
&\delta(f_*(A_1), f_*(A_2)) \\
&\quad = \sup_{u \in S^{n-1}} \left| \varrho_{f(A_1)}(u) - \varrho_{f(A_2)}(u) \right| \\
&\quad = \sup_{u \in S^{n-1}} \left| \varrho_{A_1}\big(f^{-1}(u)\big) - \varrho_{A_2}\big(f^{-1}(u)\big) \right| \\
&\quad = \sup_u \frac{1}{\| f^{-1}(u) \|} \left| \varrho_{A_1}\!\left(\frac{f^{-1}(u)}{\| f^{-1}(u) \|} \right) - \varrho_{A_2}\!\left(\frac{f^{-1}(u)}{\| f^{-1}(u) \|} \right) \right| \\
&\quad \le \sup \frac{1}{\| f^{-1}(u) \|} \cdot \delta(A_1, A_2),
\end{aligned}$$

where $\sup(1/\|f^{-1}(u)\|) < \infty$, because f^{-1} is a bijection, $f(0) = 0$, and S^{n-1} is compact.

The second assertion is obvious. ∎

2. RADIAL QUOTIENT STAR BODIES

We shall consider the following family of maps of R^n onto itself.

DEFINITION 2.1. Let $C \in \mathscr{S}_1^n$. We define $p_C\colon \mathrm{R}^n \to \mathrm{R}^n$ by the formula

$$p_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ (1 - \varrho_C(x)) \cdot x, & \text{if } x \in \mathrm{R}^n \setminus \operatorname{int} C. \end{cases}$$

PROPOSITION 2.2. *For every $C \in \mathscr{S}_1^n$:*

(i) *p_C is a quotient map;*

(ii) *if A is a star body containing C and $A \neq C$, then $p_C(A) \in \mathscr{S}_0^n$.*

In view of Proposition 2.2, we shall refer to p_C as a *radial quotient map* and to $p_C(A)$ as a *radial quotient star body of A*. We shall use the symbol A/C also for $p_C(A)$.

Let us note the following.

PROPOSITION 2.3. *If $A, C \in \mathscr{S}_1^n$ and $C \subset A$, then*

$$\varrho(A/C, x) = \varrho(A, x) - \varrho(C, x)$$

for every $x \in \mathrm{R}^n \setminus \{0\}$.

Proof. It suffices to prove the assertion for $\varrho|S^{n-1}$. Let $u \in S^{n-1}$ and $x_0 \in \operatorname{bd} A \cap \operatorname{pos} u$. Then

$$\varrho(p_C(A), u) = \| p_C(x_0) \| = (1 - \varrho_C(x_0)) \cdot \|x_0\|$$
$$= \|x_0\| - \varrho_C\left(\frac{x_0}{\|x_0\|}\right) = \varrho_A(u) - \varrho_C(u).$$

∎

The following statement is a direct consequence of Proposition 2.3.

COROLLARY 2.4. *The function $\Phi\colon \{(A, C) \in \mathscr{S}_1^n \times \mathscr{S}_1^n \mid C \subset A\} \to \mathscr{S}_1^n$ defined by*

$$\Phi(A, C) = A/C \tag{2.1}$$

is continuous with respect to the radial metric δ.

For a given $C \in \mathscr{S}_1^n$, let

$$\mathscr{S}_C^n := \{A \in \mathscr{S}_1^n \,|\, A \supset C\}. \tag{2.2}$$

THEOREM 2.5. *For every $C \in \mathscr{S}_1^n$, there exists a homotopy*

$$(\psi_t\colon \mathrm{R}^n \to \mathrm{R}^n)_{t \in [0,1]}$$

such that $\psi_0 = \mathrm{id}_{\mathrm{R}^n}$, $\psi_1 = p_C$, *and for every* $t \in [0,1]$, *the induced map* $(\psi_t)_*\colon \mathscr{S}^n_C \to \mathscr{S}^n_1$ *is an isometry with respect to* δ.

Proof. We define $\psi_t\colon \mathrm{R}^n \to \mathrm{R}^n$ by the formula

$$\psi_t(x) = \begin{cases} (1-t)\cdot x, & \text{if } x \in C, \\ (1 - t\varrho_C(x))\cdot x, & \text{if } x \notin C. \end{cases}$$

Evidently, $\psi\colon \mathrm{R}^n \times [0,1] \ni (x,t) \mapsto \psi_t(x) \in \mathrm{R}^n$ is continuous,

$$\psi(x,0) = x, \quad \text{and} \quad \psi(x,1) = p_C(x).$$

Let $t \in [0,1]$ and $A_1, A_2 \in \mathscr{S}^n_C$. We shall show that

$$\delta(A_1, A_2) = \delta(\psi_t(A_1), \psi_t(A_2)). \tag{2.3}$$

If $u \in S^{n-1}$, $i \in \{1,2\}$, and $a_i \in \mathrm{bd}\, A_i \cap \mathrm{pos}\, u$, then $\varrho(A_i, u) = \|a_i\|$ and

$$\varrho(\psi_t(A_i), u) = \|\psi_t(a_i)\| = \begin{cases} (1-t)\|a_i\|, & \text{if } a_i \in C, \\ (1 - t\varrho_C(a_i))\|a_i\|, & \text{if } a_i \notin C. \end{cases}$$

Since $(1 - t\varrho_C(a_i))\|a_i\| = \|a_i\| - t\varrho_C(u)$ and $a_1, a_2 \in C$ implies $a_1 = a_2$, by easy calculation it follows that

$$|\varrho(\psi_t(A_1), u) - \varrho(\psi_t(A_2), u)| = \begin{cases} 0, & \text{if } a_1, a_2 \in C, \\ \big|\|a_1\| - \|a_2\|\big|, & \text{otherwise}. \end{cases}$$

Thus,

$$\delta(\psi_t(a_1), \psi_t(A_2)) = \sup_{u \in S^{n-1}} |\varrho(A_1, u) - \varrho(A_2, u)| = \delta(A_1, A_2),$$

which proves (2.3). ∎

Let us observe that for every star map f the induced map f_* is a homomorphism with respect to the operation Φ defined by (2.1):

PROPOSITION 2.6. *For every* $f \in \mathrm{GS}(n)$ *and* $A, C \in \mathscr{S}^n_1$ *with* $C \subset A$,

$$f(A/C) = f(A)/f(C).$$

Proof. Since f is positively homogeneous and $\varrho_{f(C)}(f(x)) = \varrho_C(x)$, by Definition 2.1 it follows that

$$fp_C(x) = p_{f(C)}(f(x)) \quad \text{for every } x \in \mathrm{R}^n.$$

∎

The operation Φ is also invariant under the function $I_1\colon \mathscr{S}^n_1 \to \mathscr{S}^n_1$, the first of $I_1, \dots, I_{n-1}$, which are defined as follows: For every $u \in S^{n-1}$,

$$\varrho(I_k A, u) := \tilde{V}_{k,n-1}(A \cap u^{\perp}),$$

where

$$\tilde{V}_{k,n-1}(A \cap u^{\perp}) = \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \varrho(A, v)^k \, d\sigma(v)$$

for any $k \in \{1, \dots, n-1\}$ (see [4] or [7]). If $A \in \mathscr{S}_0^n$, then $I_k A$ is called the *intersection body of order* k of the star body A.

THEOREM 2.7. *For every $A, C \in \mathscr{S}_1^n$ with $C \subset A$,*

$$I_1(A/C) = I_1 A / I_1 C.$$

Proof. By Proposition 2.3, for every affine subspace E in R^n,

$$(A/C) \cap E = (A \cap E)/(C \cap E). \tag{2.4}$$

Let $u \in S^{n-1}$. Using Proposition 2.3 combined with (2.4) for $E = u^{\perp}$, we obtain

$$\varrho(I_1(A/C), u) = \varrho(I_1 A / I_1 C, u).$$

∎

As was noticed in [4, Note 8.3, p. 305], for any $k \in \{1, \dots, n-1\}$, a set A is the intersection body of order k of a star body if and only if A is the intersection body of a star body. Thus Theorem 2.7 yields the following corollary:

COROLLARY 2.8. *Let $A, C \in \mathscr{S}_1^n$ and $C \subset A$. If A and C are the intersection bodies of some star bodies, then so is A/C.*

Corollary 2.8 can be useful in geometric tomography, where the intersection bodies play an essential role. See, for instance, Theorem 8.2.8 of [4], which gives an answer to the question whether

$$V_{n-1}(A_1 \cap u^{\perp}) \leq V_{n-1}(A_2 \cap u^{\perp}) \quad \text{for every } u \in S^{n-1}$$

implies $V_n(A_1) \leq V_n(A_2)$. According to that theorem, the answer is positive when A_1 is the intersection body of a star body.

3. CATEGORICAL APPROACH. DUALITY OF STAR BODIES

Let $\mathbf{St}^n$ be the category whose objects are star bodies in R^n:

$$\mathrm{Ob}\,\mathbf{St}^n = \mathscr{S}_0^n,$$

and for every $A_1, A_2 \in \mathscr{S}_0^n$ the set of morphisms, $\mathbf{St}^n(A_1, A_2)$, consists of star maps sending A_1 to A_2. Furthermore, let $\mathbf{St}_2^n$ be the category with the class of objects

$$\mathrm{Ob}\,\mathbf{St}_2^n = \left\{(A, C) \in \mathscr{S}_0^n \times \mathscr{S}_0^n \middle| C \subset A \neq C\right\}$$

and with morphisms being suitable star maps. The function Φ: $\mathrm{Ob}\,\mathbf{St}_2^n \to \mathscr{S}_0^n$ defined by (2.1) can be extended to a functor:

PROPOSITION 3.1. *The formulae*

$$\Phi(A, C) := A/C \quad \text{for}\, A, C \in \mathscr{S}_0^n$$

and

$$\Phi(f) := \hat{f} \quad \text{for}\, f \in \mathbf{St}_2^n((A_1, C_1), (A_2, C_2))$$

[*see* (0.2)] *define a covariant functor* Φ: $\mathbf{St}_2^n \to \mathbf{St}^n$, *such that any morphism and its image under* Φ *are restrictions of the same star map*.

Proof. In view of Theorem 0.1, it suffices to show that $\hat{f} = f$ for every $f \in \mathrm{GS}(n)$. It is easy to check that, for every $x \in \mathrm{R}^n \setminus \{0\}$,

$$p_C^{-1}(x) = (1 + \varrho_C(x)) \cdot x$$

and

$$p_C(\beta x) = (\beta - \varrho_C(x)) \cdot x$$

for every $\beta \in \mathrm{R}$. Thus,

$$\begin{aligned}\hat{f}(x) &= p_{f(C)} f p_C^{-1}(x) = p_{f(C)} f\big((1 + \varrho_C(x)) \cdot x\big) \\ &= p_{f(C)}\big((1 + \varrho_C(x)) f(x)\big) = \big(1 + \varrho_C(x) - \varrho_{f(C)}(f(x))\big) f(x) \\ &= f(x).\end{aligned}$$

∎

Given a category $\mathbf{C}$ of star bodies, it is natural to look for a *duality* in this category, that is, a functor from $\mathbf{C}$ to itself being an involution (compare [11]). In what follows we shall restrict our consideration to star bodies with positive radial function (i.e., with the origin in the interior). Let $\mathbf{St}^n_+$ be the corresponding full subcategory of $\mathbf{St}^n$.

Let i be the inversion of the one-point compactification $\overline{\mathrm{R}}^n$ of R^n, with respect to S^{n-1}:

$$i(x) := \frac{x}{\|x\|^2} \quad \text{for } x \in \mathrm{R}^n \setminus \{0\}. \tag{3.1}$$

DEFINITION 3.2. For every object A of $\mathbf{St}^n_+$,

$$A^\circ = \mathrm{cl}(\mathrm{R}^n \setminus i(A));$$

for every $f \in \mathrm{GS}(n)$,

$$f^\circ := ifi \,|\, \mathrm{R}^n.$$

Let us note the following.

PROPOSITION 3.3. *Let $A \in \mathscr{S}^n_0$ and $0 \in \mathrm{int}\, A$. Then $A^\circ \in \mathscr{S}^n_0$ and for every $u \in S^{n-1}$,*

$$\varrho(A^\circ, u) = \frac{1}{\varrho(A,u)}.$$

Proof. By Definition 3.2, A° is a star body and

$$\varrho(A^\circ, u) = \|a\|,$$

where $a \in \mathrm{bd}(\mathrm{R}^n \setminus i(A)) \cap \mathrm{pos}\, u$. Since $\mathrm{bd}(\mathrm{R}^n \setminus A) = i(\mathrm{bd}\, A)$, it follows that $i(a) \in \mathrm{bd}\, A \cap \mathrm{pos}\, u$, and thus

$$\|i(a)\| = \varrho(A,u).$$

By (3.1), this completes the proof. ∎

THEOREM 3.4. *The pair of functions $A \mapsto A^\circ$, $f \mapsto f^\circ$ is a covariant functor from $\mathbf{St}^n_+$ to itself such that*

$$A^{\circ\circ} = A \quad \textit{and} \quad f^{\circ\circ} = f \tag{3.2}$$

for every A and f.

Moreover,

$$A \subset B \quad \Rightarrow \quad B^\circ \subset A^\circ. \tag{3.3}$$

Proof. By Proposition 3.3, if A is a star body with 0 in the interior, then so is A°. Let $f \in \mathrm{GS}(n)$ and $f(A) = B$. Then

$$f^\circ \in \mathrm{GS}(n), \tag{3.4}$$

because f° is a homeomorphism of R^n and for every positive α,

$$f^\circ(\alpha x) = ifi(\alpha x) = f\left(\frac{1}{\alpha} i(x)\right) = \alpha ifi(x) = \alpha f^\circ(x).$$

Moreover,

$$f^\circ(A^\circ) = B^\circ, \tag{3.5}$$

because

$$f^\circ(A^\circ) = ifi\big(\mathrm{cl}(\mathrm{R}^n \setminus i(A))\big) = \mathrm{cl}(\mathrm{R}^n \setminus if(A)) = (f(A))^\circ.$$

By (3.4) and (3.5), we have a covariant functor.

Furthermore,

$$\begin{aligned} A^{\circ\circ} &= \mathrm{cl}(\mathrm{R}^n \setminus i(A^\circ)) = \mathrm{cl}\big(\mathrm{R}^n \setminus i\big(\mathrm{cl}\big(\mathrm{R}^n \setminus i(A)\big)\big)\big) \\ &= \mathrm{R}^n \setminus i\big(\mathrm{int\,cl}(\mathrm{R}^n \setminus i(A))\big) = A, \end{aligned}$$

since f and i are homeomorphisms, $0 \in \mathrm{int}\, A$, and i is an involution. Evidently, $f^{\circ\circ} = f$. This proves (3.2). By similar arguments, (3.3) holds. ∎

By Theorem 3.4, the pair of functions $A \mapsto A^\circ$, $f \mapsto f^\circ$ is a duality. We shall refer to this functor as the *star duality*.

For a convex body A with the origin in the interior, there is a simple connection between polar dual A^* and star dual A°.

THEOREM 3.5. *If* $A \in \mathscr{K}_0^n$ *and* $0 \in \mathrm{int}\, A$, *then*

$$\tilde{V}_1(A^\circ) = \frac{\kappa_n}{2} \cdot b(A^*).$$

Proof. By Proposition 3.4 and Remark 1.7.7 in [12],

$$\varrho(A^\circ, u) = h(A^*, u) \quad \text{for every } u \in S^{n-1}. \tag{3.6}$$

Hence, by the formula (1.7.2) in [12],

$$\begin{aligned} \tilde{V}_1(A^\circ) &= \frac{1}{n}\int_{S^{n-1}} \varrho(A^\circ, u)\, d\sigma(u) = \frac{1}{n}\int_{S^{n-1}} h(A^*, u)\, d\sigma(u) \\ &= \frac{\kappa_n}{2} \cdot b(A^*). \end{aligned}$$

∎

Generally, star dual of a convex body is different from its polar dual; see Fig. 1.

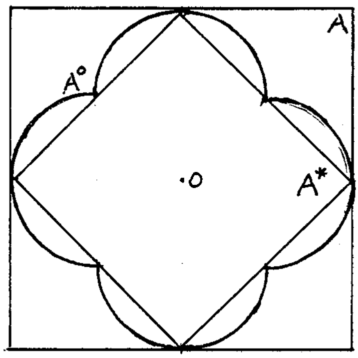

FIG. 1. A, big square; A^*, small square; A°, flower.

THEOREM 3.6. *Let $n \geq 2$. For every $A \in \mathscr{K}_0^n$ with $0 \in \operatorname{int} A$, the following are equivalent*:

(i) $A^\circ = A^*$;

(ii) *A is a centered ball.*

Proof. (ii) $\Rightarrow$ (i) is obvious.

Assume (i). Then, by (3.6),

$$\varrho(A^*, u) = h(A^*, u) \quad \text{for every } u \in S^{n-1}$$

and thus

$$\operatorname{pos} u \cap \operatorname{bd} A^* \subset H(A^*, u).$$

As a consequence, bd A^* is of the class C^1.

Since $E \cap A^\circ = (E \cap A)^\circ$ for every two-dimensional linear subspace E, without any loss of generality we may assume that $n = 2$.

Let r: R $\to$ bd A^* be the parametrization of bd A^* in polar coordinates,

$$r(t) = \bar{\varrho}(t) \cdot u(t),$$

where $\bar{\varrho}(t) = \varrho(A^*, u(t))$ and $u(t) = (\cos t, \sin t)$. Then $\langle r(t), r'(t) \rangle = 0$ for every t; i.e., $\bar{\varrho} \cdot \bar{\varrho}' = 0$. Hence $\bar{\varrho}' = 0$ and thus ϱ_{A^*} is constant, and so is ϱ_A. This completes the proof. ∎

The following statement is an analogue of the conjecture (7.4.33) in [12], concerning the polar duality of convex bodies.

THEOREM 3.7. *Let A be a star body in* R^n *with* $0 \in \operatorname{int} A$. *For any* $p, q > 0$ *with* $1/p + 1/q = 1$,

$$\tilde{V}_p(A^\circ) \cdot \tilde{V}_q(A)^{p-1} \geq \kappa_n^p. \tag{3.7}$$

In particular,

$$\tilde{V}_2(A^\circ) \cdot \tilde{V}_2(A) \geq \kappa_n^2. \tag{3.8}$$

The equality in (3.7) *or* (3.8) *holds if and only if A is a centered ball.*

Proof. Since $\varrho_{A^\circ} \cdot \varrho_A = 1$ (see Proposition 3.3), by the Hölder inequality,

$$\begin{aligned} n\kappa_n &= \sigma(S^{n-1}) = \int_{S^{n-1}} \varrho_{A^\circ} \varrho_A \, d\sigma \\ &\leq \left(\int_{S^{n-1}} (\varrho_{A_0})^p \, d\sigma \right)^{1/p} \cdot \left(\int_{S^{n-1}} (\varrho_A)^q \, d\sigma \right)^{1/q} \\ &= \left(n\tilde{V}_p(A^\circ) \right)^{1/p} \cdot \left(n\tilde{V}_q(A) \right)^{1/q} = n\tilde{V}_p(A^\circ)^{1/p} \cdot \tilde{V}_q(A)^{(p-1)/p}. \end{aligned}$$

This implies (3.7) (and 3.8).

The equality holds if and only if there exists $\alpha > 0$ such that $\varrho_{A^\circ} = \alpha \varrho_A$. This happens if and only if A is a centered ball. ∎

Finally, let us return to the notion of intersection body of order k. It has been defined only for $k > 0$ (see [4, Note 8.3]), but its definition can be automatically extended over $k < 0$. The following statement is evident:

3.8. *For every star body A with* $0 \in \operatorname{int} A$,

$$I_{-k} A = I_k(A^\circ).$$

There is a relationship between star duality and the operation I_k:

THEOREM 3.9. *Let* $|k| \in \{1, \ldots, n-1\}$, $A \in \mathscr{S}_0^n$, *and* $0 \in \operatorname{int} A$. *Then*

$$\kappa_{n-1}^2 \cdot (I_k A)^\circ \subset I_k(A^\circ) \tag{3.9}$$

and equality holds if and only if A is a centered ball.

Proof. Let $X := I_k A$ and $Y := I_k(A^\circ)$. Then (3.9) is equivalent to the condition

$$\kappa_{n-1}^2 \varrho(X^\circ, u) \leq \varrho(Y, u) \quad \text{for every } u \in S^{n-1},$$

which, in view of Proposition 3.3, can be rewritten as

$$\kappa_{n-1}^2 \leq \varrho_X \cdot \varrho_Y. \tag{3.10}$$

Using the definition of I_k, the Hölder inequality, and Proposition 3.3, we obtain

$$\begin{aligned}
&\varrho_X(u)\cdot\varrho_Y(u)\\
&\quad=\frac{1}{(n-1)^2}\int_{S^{n-1}\cap u^{\perp}}\varrho_A(v)^k\,d\sigma(v)\cdot\int_{S^{n-1}\cap u^{\perp}}\varrho_{A^\circ}(v)^k\,d\sigma(v)\\
&\quad\geq\frac{1}{(n-1)^2}\left(\int_{S^{n-1}\cap u^{\perp}}1\,d\sigma(v)\right)^2=\kappa_{n-1}^2.
\end{aligned}$$

Thus (3.10) is satisfied.

The equality in (3.9) holds if and only if it holds in the Hölder inequality for the functions $(\varrho_A)^{k/2}$ and $(\varrho_{A^\circ})^{k/2}$; hence, by Proposition 3.3, it holds if and only if ϱ_A is constant. This completes the proof. ∎

Let us observe that, in view of Proposition 3.3, if I_k is replaced by the proportional function $J_k := (\kappa_{n-1})^{-1}I_k$, then condition (3.9) can be expressed in the more elegant form

$$(J_kA)^\circ\subset J_k(A^\circ). \tag{3.9$'$}$$

4. FINAL REMARKS

A. *The Classification Problem*

To every $A\in\mathscr{S}_0^n$ we assign the subset S_A of the unit sphere:

$$S_A=\left\{u\in S^{n-1}\,\middle|\,\varrho_A(u)>0\right\}. \tag{4.1}$$

THEOREM 4.1. *Let $A_1, A_2\in\mathscr{S}_0^n$. If there exists $f\in \mathrm{GS}(n)$ such that $f(A_1)=A_2$, then S_{A_1} is homeomorphic to S_{A_2}.*

Proof. If $S_{A_1}=S^{n-1}$, then also $S_{A_2}=S^{n-1}$ and the assertion is trivial. Let $S^{n-1}\setminus S_{A_1}\neq\varnothing$ and let $g_i\colon S_{A_i}\to \operatorname{bd}A_i\setminus\{0\}$ be the central projection for $i=1,2$:

$$g_i(u)=\varrho_{A_i}(u)\cdot u.$$

Clearly, g_1 and g_2 are homeomorphisms. Thus, setting

$$h(u)=(g_2)^{-1}fg_1(u)\quad\text{for } u\in S_{A_1},$$

we obtain a homeomorphism $h\colon S_{A_1}\to S_{A_2}$, as required. ∎

The following problem is open.

Problem 1. Is the existence of a homeomorphism of S_{A_1} onto S_{A_2} sufficient for A_1, A_2 to be star equivalent?

B. *Topological Properties of Star Bodies*

While, evidently, star bodies in the sense of [12] are homeomorphic to B^n, the elements of $\mathscr{S}_0^n$ are retracts of B^n and thus they are absolute retracts (see [1]):

PROPOSITION 4.2. *If $A \in \mathscr{S}_0^n$, then A is an absolute retract.*

Proof. Of course, without any loss of generality we may assume that $A \subset B^n$. Let us define r: $B^n \to A$ by the formula

$$r(x) = \begin{cases} x\varrho_A(x), & \text{if } x \in B^n \setminus A, \\ x, & \text{if } x \in A. \end{cases}$$

Then r is continuous, because

$$x\varrho_A(x) = x \quad \text{for } x \in \text{bd } A \setminus \{0\},$$

and

$$x\varrho_A(x) = u\varrho_A(u) \quad \text{for } u = \frac{x}{\|x\|},$$

where $\varrho_A(u) \to 0$ if $x \to 0 \in \text{bd } A$.

Thus, r is a retraction, since $r|A = \text{id}$. This completes the proof. ∎

C. *Convex Quotient Bodies*

It is easy to see that, for convex bodies A, C with $0 \in C \subset A$, the quotient star body A/C generally need not be convex, but sometimes is convex; for instance, homothetic convex bodies have convex quotient bodies:

PROPOSITION 4.3. *Let $A, C \in \mathscr{K}_0^n$ and $C \subset A$. If $A = \lambda C$ for some $\lambda > 1$, then $A/C \in \mathscr{K}_0^n$.*

Proof. Let $f(x) = \lambda x$ for every $x \in \mathrm{R}^n$. Then $f \in \mathrm{GS}(n)$; thus, by Proposition 1.2,

$$\varrho_A(\lambda x) = \varrho_{f(C)}(f(x)) = \varrho_C(x)$$

for every $x \neq 0$. Since ϱ_A is homogeneous of degree -1, it follows that $(1/\lambda)\varrho_A(x) = \varrho_C(x)$. This, together with Proposition 2.3, implies

$$\varrho_{A/C}(x) = (\lambda - 1)\varrho_C(x).$$

Thus, $\varrho_{A/C} = \varrho_{(\lambda-1)C}$ and consequently $A/C = (\lambda - 1)C$. Hence $A/C \in \mathscr{K}_0^n$. ∎

Problem 2. What can be said about the pairs (A, C) in $\mathscr{K}_0^n$ with convex A/C?

A contribution to this problem will be the subject of a separate paper, by Tomasz Żukowski.

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